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Standard theories of fuzzy sets with the law $(\mu \wedge \sigma')' = \sigma \vee (\mu' \wedge \sigma')$ ☆

E. Trillas [a,*], C. Alsina [b]

[a] *Departmento de Inteligencia Artificial, Universidad Politécnica de Madrid, Boadilla del Monte, Madrid 28660, Spain*

[b] *Secció de Matemàtiques i Informàtica, ETSAB, Universitat Politècnica de Catalunya, Spain*

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Abstract

It is known that, in standard theories of fuzzy sets $([0,1]^X, \wedge, \vee, ')$, the law $(\mu \wedge \sigma')' = \sigma \vee (\mu' \wedge \sigma')$ does not hold if $\wedge$ and $\vee$ are dual. It is also known that considering, in both sides of this formula, different t-norms and negation functions, there are uncountable many solutions of the equivalent functional equation $N_1(T_1(a, N_2(b))) = S(b, T_2(N_3(a), N_4(b)))$ in the unknowns N_1, N_2, N_3, N_4, T_1, T_2, and S. Nevertheless, since the simplest situation in which $N_1 = N_2 = N_3 = N_4$, $T_1 = T_2$, remained open, this paper is devoted to completely solve this particular case. That is, to study in which standard theories of fuzzy sets $([0,1]^{[0,1]}, T, S, N)$ the above law holds. The solution is that the law only holds in the theories isomorphic to $([0,1]^X, \mathrm{Prod}, W^*, 1 - id_{[0,1]})$. This opens the door to consider nondual standard theories of fuzzy sets, a field until today largely ignored.

Keywords: Fuzzy logic; Logical laws; t-norms; t-conorms; Negations

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* Corresponding author. Fax: +34-91-352-4819.
E-mail address: etrillas@fi.upm.es (E. Trillas).

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1. Introduction

1.1. In [1] Elkan considered the logical law $\neg(p \wedge \neg q) = q \vee (\neg p \wedge \neg q)$, typical of boolean algebras (see [4]), in a particular multiple-valued logic not necessarily having true and false statements. This law was used there to show a pretended "collapse" of fuzzy logic into bivaluate logic but, of course, the only conclusion was the nonsurprising fact that the particular multiple-valued logic taken into account was not consistent with the given law (see [5]).

Indeed the result in [1] says nothing in fuzzy logic and nothing on both orthomodular lattices and De Morgan algebras, that collapse into boolean algebras when the law $\neg(p \wedge \neg q) = q \vee (\neg p \wedge \neg q)$ is imposed to them. Notwithstanding, and because of these facts, it is interesting to known if there are nonlattice structures keeping the above law in the case in which $p = \mu$ and $q = \sigma$ are fuzzy sets in a theory $([0,1]^X, \wedge, \vee, ')$ with $\wedge \neq \min$ and $\vee \neq \max$, (see [2,6]). Of course, if these theories do exist, there will be some "cost" in the sense of loosing some other laws, as it happened in several cases like that of keeping the law of noncontradiction, whose main cost is distributivity.

1.2. In [2] we studied the logical law

$$(\mu \wedge \sigma')' = \sigma \vee (\mu' \wedge \sigma') \tag{$*$}$$

(see [1]), when μ and σ are in $[0,1]^X$ and the connectives $\wedge$ and $'$ are represented either by different t-norms or different strong-negation functions. We considered the result for the case of a single standard theory of fuzzy sets $([0,1]^X, T, S, N)$ in which $\mu \wedge \sigma = T \circ (\mu \times \sigma)$, $\mu \vee \sigma = S \circ (\mu \times \sigma)$, and $\mu' = N \circ \mu$, with T a continuous t-norm, S a continuous t-conorm, and N a strong-negation function. That result says that $(*)$ implies that T, S and N cannot be linked by $T = N \circ S \circ (N \times N)$, that is, T and S cannot be N-dual.

In [2] it is also shown that $T \neq N \circ S \circ (N \times N)$ is not a sufficient condition for the validity of $(*)$. The counterexample is given by $T = \mathrm{Min}$, $S = W^*$ and $N = 1 - id_{[0,1]}$.

Eq. $(*)$ is obviously equivalent to the functional equation

$$N(T(a, N(b))) = S(b, T(N(a), N(b))) \tag{$**$}$$

for all a, b in $[0,1]$.

In next section, we will prove that this Eq. $(**)$ in the unknowns T, S and N holds if and only if $T = \mathrm{Prod}_\varphi = \varphi^{-1} \circ \mathrm{Prod} \circ (\varphi \times \varphi)$, $S = W^*_\varphi = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$ and $N = N_\varphi = \varphi^{-1} \circ (1 - id_{[0,1]}) \circ \varphi$, where the bijective function $\varphi : [0,1] \to [0,1]$ is strictly increasing and verifying $\varphi(0) = 0$ and $\varphi(1) = 1$, i.e., φ is an order-automorphism of the unit interval.

Lemma 1. *Any theory* $([0,1]^X, \mathrm{Prod}_\varphi, W^*_\varphi, N_\varphi)$ *is isomorphic with the theory* $([0,1]^X, \mathrm{Prod}, W^*, 1 - id_{[0,1]})$.

Proof. The mapping $\phi : [0,1]^X \to [0,1]^X$, $\phi(\mu) = \varphi^{-1} \circ \mu$ is one-to-one and verifies:

- $\mathrm{Prod}_\varphi \circ (\phi(\mu) \times \phi(\sigma)) = \varphi^{-1} \circ \mathrm{Prod} \circ (\varphi(\phi(\mu)) \times \varphi(\phi(\sigma))) = \varphi^{-1} \circ \mathrm{Prod} \circ (\mu \times \sigma) = \phi(\mathrm{Prod} \circ (\mu \times \sigma))$
- $W^*_\varphi \circ (\phi(\mu) \times \phi(\sigma)) = \varphi^{-1} \circ W^* \circ (\varphi(\phi(\mu)) \times \varphi(\phi(\sigma))) = \varphi^{-1} \circ W^* \circ (\mu \times \sigma) = \phi(W^*(\mu \times \sigma))$
- $N_\varphi \circ \phi(\mu) = \varphi^{-1} \circ (1 - id_{[0,1]}) \circ \varphi \circ \varphi^{-1} \circ \mu = \varphi^{-1}(1 - id_{[0,1]}) \circ \mu = \phi((1 - id_{[0,1]}) \circ \mu)$

Hence, ϕ is an isomorphism between $([0,1]^X, \mathrm{Prod}_\varphi, W^*_\varphi, N_\varphi)$ and $([0,1]^X, \mathrm{Prod}, W^*, 1 - id_{[0,1]})$. $\square$

Our aim is to show the following:

Theorem 1. *The only standard theories of fuzzy sets* $([0,1]^X, T, S, N)$ *in which* $(*)$ *is a law are those isomorphic to the theory* $([0,1]^X, \mathrm{Prod}, W^*, 1 - id_{[0,1]})$.

Consequently, the standard theories of fuzzy sets with the law $(*)$ verify, in addition to all laws that are common to standard theories, the law of excluded-middle $\mu \vee \mu' = \mu_X$ and the law of von Neumann $\mu = (\mu \wedge \sigma) \vee (\mu \wedge \sigma')$. Obviously, they do not verify the laws of De Morgan and the law of non-contradiction.

2. A necessary and sufficient condition for $(**)$

Our aim in this section is to solve $(**)$ completely.

Lemma 2. *Eq.* $(**)$ *holds in any theory* $([0,1]^X, \mathrm{Prod}_\varphi, W^*_\varphi, N_\varphi)$.

Proof. After Lemma 1 it is enough to check $(**)$ with $T = \mathrm{Prod}$ $S = W^*$ and $N = 1 - id_{[0,1]}$ In fact, $N(T(a, N(b))) = 1 - a(1-b) = 1 - a + ab$, and $S(b, T(N(a), N(b))) = W^*(b, (1-a)(1-b)) = \mathrm{Min}(1, b + (1-a)(1-b)) = \mathrm{Min}(1, 1 - a + ab) = 1 - a + ab$ Hence, sufficiency is proven. $\square$

Lemma 3. *If* $(**)$ *holds in* $([0,1]^X, T, S, N)$, *it is* $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$, $N_\varphi \leqslant N$ *and* $T = \psi^{-1} \circ \mathrm{Prod} \circ (\psi \times \psi)$.

Proof. With $a = 0$, $(**)$ it yields $1 = S(b, N(b))$ for all b in $[0, 1]$.

Hence, $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$ and $N_\varphi \leqslant N$ for some order-automorphism φ of the unit interval. If T has an idempotent $a_0 \in (0, 1)$ it results $S(N(a_0), T(a_0, a_0)) = N(T(N(a_0), a_0))$, with $b = N(a_0), a = a_0$ in $(**)$. Hence, $1 = S(N(a_0), a_0) = N(\mathrm{Min}(a_0, a_0)) = \mathrm{Max}(a_0, N(a_0))$ which implies $a_0 \in \{0, 1\}$ which is absurd. Consequently, T is neither Min nor an ordinal-sum. If $T = \psi^{-1} \circ W \circ (\psi \times \psi)$, it will be $T(a, N_\psi(a)) = 0$ for all a in $[0, 1]$.

Changing b by $N(b)$ and a by $N(a)$, (**) gives $S(N(b), T(a, b)) = N(T(N(a), b))$ so, with $b = N_\psi(x)$ and $a = x$, yields to $S(N(N_\psi(x)), T(x, N_\psi(x))) = N(T(N(x), N_\psi(x)))$ or $N(N_\psi(x)) = N(T(N(x), N_\psi(x)))$, and finally $N_\psi(x) = T(N(x), N_\psi(x))$, for all x in $[0, 1]$, which is equivalent to $1 - \psi(x) = \mathrm{Max}(0, \psi(N(x)) - \psi(x))$. This last equation is contradictory since, taking $x_0 = N(x_0) \in (0, 1)$, it would yield $1 = \psi(x_0)$.

Hence T should be in the family of Prod : $T = \psi^{-1} \circ \mathrm{Prod} \circ (\psi \times \psi)$, for some order-automorphism ψ of the unit interval. $\square$

Lemma 4. $N = N_\varphi$.

Proof. If there would exist x_0 in $(0, 1)$ such that $N(x_0) > N_\varphi(x_0) = \varphi^{-1}(1 - \varphi(x_0))$, take z_0 such that $N(x_0) > z_0 > N_\varphi(x_0)$, and the function $F : [0, 1] \to [0, N(x_0)]$, defined by $F(x) = T(x, N(x_0))$, which is continuous and strictly increasing. Hence, there is $\alpha \in (0, 1)$ such that $T(\alpha, N(x_0)) = z_0$, and $S(x_0, T(\alpha, N(x_0))) = N(T(N(\alpha), N(x_0)))$, $1 = S(x_0, z_0) = N(T(N(\alpha), N(x_0)))$ which implies the absurd $0 = T(N(\alpha), N(x_0))$. $\square$

Lemma 5. $\varphi = \psi$.

Proof. From $(**)$, with $S = W^*_\varphi$, $T = \mathrm{Prod}_\psi$, and $N = N_\varphi$, it results: $\varphi^{-1}[1 - \varphi\psi^{-1}(\psi(a) \cdot \psi(\varphi^{-1}(1 - \varphi(b))))] = \varphi^{-1}[\varphi(b) + \varphi\psi^{-1}(\psi(N(a)) \cdot \psi(N(b)))]$ for all a, b in $(0, 1)$, because for $a, b \in \{0, 1\}$ Eq. $(**)$ always holds.

Take $\varphi(a) = \mu$, $\varphi(b) = v$, and $\varphi_0\psi^{-1} = f$. It results: $1 - f^{-1}(f(u) \cdot f(1 - v)) = v + f^{-1}(f(1 - u) \cdot f(1 - v))$, and $F(a, b) = f^{-1}(f(a) \cdot f(b))$ is a t-norm such that, with $w = 1 - v$, verifies $w = F(u, w) + F(1 - u, w) = W^*(F(u, v), F(1 - u, w))$.

This equation was studied in [3], and the solution is $F = \mathrm{Prod}$. Hence, $f = id_{[0,1]}$ and $\varphi = \psi$. $\square$

Theorem 2. *The only solutions for the functional equation* $N(T(a, N(b)) = S(b, T(N(a), N(b))))$, *for all* a, b *in* $[0, 1]$, *where* T *is a continuous t-norm*, S *is a continuous t-conorm*, *and* N *is a strong-negation function*, *are those given by* $T = \varphi^{-1} \circ \mathrm{Prod} \circ (\varphi \times \varphi)$ $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$, *and* $N = \varphi^{-1} \circ (1 - \varphi)$, *for all order automorphisms* φ *of the unit interval* $[0, 1]$.

Proof. It follows directly from Lemmas 2–5. □

Examples

1. With $T = \mathrm{Prod}, S = W^*$ and $N = 1 - id_{[0,1]}$, it is:
   - $N(T(a, N(b))) = 1 - a + ab$; Reichenbach implication.
   - $S(b, T(N(a), N(b))) = W^*(b, (1-a)(1-b)) = \min(1, 1 - a + ab) = 1 - a + ab$; Reichenbach implication.
2. With $T = W$, $S = W^*$ and $N = 1 - id_{[0,1]}$, it is:
   - $N(T(a, N(b))) = 1 - W(a, 1-b) = \min(1, 1 - a + b)$; Łukasiewicz implication.
   - $S(b, T(N(a), N(b))) = W^*(b, W(1-a, 1-b)) = \min(1, \max(b, 1-a)) = \max(1-a, b)$; Kleene–Dienes implication.

   Since $\max(1-a, b) < \min(1, 1-a+b)$, in the theory $([0,1]^X, W, W^*, 1 - id_{[0,1]})$ it is not $(\mu \wedge \sigma')' = \sigma \vee (\mu' \wedge \sigma')$, but only $(\mu \wedge \sigma')' \leqslant \sigma \vee (\mu' \wedge \sigma')$.
3. With $T = \mathrm{Prod}$, $S = \mathrm{Prod}^*$, $N = 1 - id_{[0,1]}$, it is:
   - $N(T(a, N(b)) = 1 - a + ab$; Reichenbach implication.
   - $S(b, T(N(a), N(b))) = \mathrm{Prod}^*(b, (1-a)(1-b)) = 1 - a - b + 2ab + b^2 - ab^2$.

   but $1 - a + ab = 1 - a - b + 2ab + b^2 - ab^2$, equivalent to $b[a(1-b) + b(1-a)] = 0$, holds if and only if a, b are in $\{0, 1\}$.

3. Conclusion

3.1. In very general frameworks (see [2]), and like with other classical logical laws, the validity of $(*)$ in fuzzy set theories has some "legal" cost. That is, it can be reached at the cost of loosing some other laws of classical logic.

As this paper shows, the cost of having the law $(*)$ in a standard theory $([0,1]^X, T, S, N)$ is the loss of the laws of idempotency $(\mu \wedge \mu = \mu, \mu \vee \mu = \mu)$, distributivity $(\mu \wedge (\sigma \vee \delta) = (\mu \wedge \sigma) \vee (\mu \wedge \delta), \mu \vee (\sigma \wedge \delta) = (\mu \vee \sigma) \wedge (\mu \vee \delta))$, duality $\mu' \vee \sigma' = (\mu \wedge \sigma)', \mu' \wedge \sigma' = (\mu \vee \sigma)')$, and noncontradiction $(\mu \wedge \mu' = \mu_\phi)$.

If this loss cannot be considered as a surprise, what it is perhaps surprising is the conservation of both the law of excluded-middle $(\mu \vee \mu' = \mu_X)$ and von Neumann's law $(\mu = (\mu \wedge \sigma) \vee (\mu \wedge \sigma'))$. In any case, the statement in [2] that there are uncountable many theories of fuzzy sets with the law $(*)$, can be now re-stated and particularized by saying that if there are uncountable many standard theories $([0,1]^X, T, S, N)$ of fuzzy sets with the law $(*)$, all of them are isomorphic to the theory $([0,1]^X, \mathrm{Prod}, W^*, 1 - id_{[0,1]})$ That is, that this last theory is the model of *a* standard theory for the validity of the law $(\mu \wedge \sigma')' = \sigma \vee (\mu' \wedge \sigma')$.

Since $N_\varphi(\mathrm{Prod}_\varphi(a, N_\varphi(b))) = \mathrm{Prod}^*_\varphi(N_\varphi(a), b)$, it should be pointed out that, when $T = \mathrm{Prod}_\varphi$ and $N = N_\varphi$, the left hand side $((\mu \wedge \sigma')')$ of $(*)$ is nothing else than the S-implication $\mathrm{Prod}^*_\varphi \circ (N_\varphi \circ \mu \times \sigma)$, that is, $(\mu \rightarrow \sigma)(x, y) = \varphi^{-1}(\varphi(\mu(x)) + \varphi(\sigma(y)) - \varphi(\mu(x)) \cdot \varphi(\sigma(y))$ that, in the case $\varphi = id_{[0,1]}$ is Reichenbach implication $\mu(x) + \sigma(y) - \mu(x) \cdot \sigma(y)$. Hence, like in boolean algebras, in the case of the theories $([0,1]^X, \mathrm{Prod}_\varphi, W^*_\varphi, N_\varphi)$ formula $(*)$ can be re-written as $\mu \rightarrow \sigma = \sigma \vee (\mu' \wedge \sigma')$.

3.2. Theorem 2 opens in approximate reasoning, a practically new subject: to study the classes of reasoning representable within nondual theories of fuzzy sets. Let us mention an interesting example about this problem.

In some countries, degrees of partial impairment are composed to a total degree by mean of Balthazard formula (see [7]) $B(a, b) = a + b - ab = \mathrm{Prod}^*(a, b)$, under which the total degree of impairment $B(a, b)$ after two partial degrees appears as "degree a or degree b". But Victor Balthazard (see [8]) conducted his reasoning in the following way: after the first impairment with degree a, the "healthy" rest $1 - a$ is affected by the second impairment b giving the new partial degree $(1 - a)b$, and then the composed degree is $B(a, b) = a + (1 - a)b = W^*(a, \mathrm{Prod}(1 - a, b))$. Hence, $B(a, b)$ can be read as "degree a or (degree $1 - a$ and degree b)" showing, better than $\mathrm{Prod}^*(a, b)$, the problems that this formula produces if, for example, the second impairment either matches the first or is not independent of it.

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